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THE CONSTRUCTION OF OPTIMUM THREE-DIMENSIONAL SHAPES WITHIN THE FRAMEWORK OF A MODEL OF LOCAL INTERACTION†

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Within the framework of the law of locality, where a force acting on a surface element of a body is a known function of the orientation of the element with respect to the direction of motion, without constraints on the form of this function and the thickness of the body, the problem of constructing the three-dimensional shape of a body of minimum drag is solved. It is shown that, for a specified base area of the body and its maximum permissible length, the problem has an infinite set of solutions. Here, all the optimum bodies are conical, and their drag is identical. The surface of these bodies is formed by combinations of surface areas of a circular cone and planes tangential to it. The angle at the tip of the circular cone is determined by the characteristics of the medium and by the velocity of motion in the constants occurring in the drag law. © 2000 Elsevier Science Ltd. All rights reserved.

Assuming that the force of action of a medium on a surface element of a body depends solely on its orientation with respect to the direction of motion, the problem of determining the three-dimensional shape of a body of minimum drag has been the subject of numerous investigations [1–8]. Even within the scope of specific drag laws, the solution of this problem by indirect methods of variational calculation has been found only with simplifying assumptions regarding the geometry of the body, where, by narrowing the category of permissible surface,s it has proved possible to reduce Euler's equation to ordinary differential equations.

Thin three-dimensional bodies of minimum drag with similar (homothetic) cross-sections have been investigated [1, 2, 4–8]. Here, either the longitudinal contour of the body was assumed to be specified [2, 5] or a distribution of the pressure and friction coefficients was selected by which the problems of determining the optimum longitudinal and transversal contours were separated [1, 4, 7, 8]. Thus, on the assumption that the pressure distribution is specified by Newton's formula, with zero friction the optimum bodies, having the specified length and base area, were found [1, 4], it being assumed that, in the base plane of the body, the minimum radius of its cross-section was also specified. According to these results, the longitudinal contour of a three-dimensional body of minimum wave drag is determined by a power law function with an exponent of 3/4, while the transversal contour has a star shape and consists of a previously specified number of symmetrical cycles. In a similar formulation, without any constraint on the body thickness, the optimum transversal contour was constructed [3] for the specified (conical) longitudinal contour.

An increase in the number of cycles of optimum starbodies monotonically reduced their drag [1, 3, 4]. With a low wave drag, these bodies had a much greater surface area than did bodies of revolution of equivalent length and base area, which, taking friction into account, could lead to a considerable increase in their total drag.

The problem of constructing the optimum transversal contour of a body was examined [2] taking friction into account (see also [5]), the pressure on the body surface being specified by Newton's formula, while the coefficient of friction was assumed to be constant. For a specified length and base area it was assumed that the body is slender and has a specified (power-law) longitudinal contour. The most important results [2, 5] in this formulation of the problem include the conclusion that its solution is possibly non-unique for values of the coefficient of friction below the 'critical', a category of various bodies was constructed with identical drag. The transversal contour of these bodies can comprise combinations of arcs of a circle and segments of straight lines tangential to it. In particular, it could be star-shaped and consist of an arbitrarily specified number of symmetrical cycles. Here, the radius of the circle to which the lines were tangential was determined by the selected shape of the longitudinal contour and by the coefficient of friction. It was shown [6] that the drag of such bodies with a known

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friction parameter is absolutely minimum. This solution became unique when the value of the coefficient of friction was greater than the critical value or when additional constraints were imposed on the geometry of the body, for example, the minimum radius of the base generator or the number of cycles in it was specified.

In a formulation of the problem, which was similar to that proposed earlier [1, 4] but taking friction into account, the optimum longitudinal and transversal contours were investigated [7] and here, as earlier [2, 5], the coefficient of friction was assumed to be constant. For thin homothetic bodies, variational problems of determining the optimum longitudinal and transversal contours were separated [7]. Here, the equation defining the optimum longitudinal contour contained a constant that was expressed in terms of an integral which depended on the transversal contour, and, conversely, the equation for the optimum transversal contour contained a constant that depended on the longitudinal contour. These previously unknown "constraint constants" could be found only by a combined solution of the problems. An analysis carried out with arbitrary values of these showed that, in the general case (including when only the base area and length of the body are specified), the optimum longitudinal contour in the vicinity of the tip is determined by a power law function with an exponent of 3/4, while the transverse contour, as earlier [1, 4], is star-shaped. The combined problem of determining the optimum longitudinal and transversal contours of a body has not been solved, but is has been noted [7] that situations are possible in which the longitudinal contour becomes conical, while the transverse contour consists of straight line segments.

In a formulation similar to that proposed earlier [7], in particular, assuming the bodies required to be thin and homothetic bodies, using a well known procedure [5, 7] a more complete analysis of the optimum three-dimensional configurations was carried out [8]. An important result [8] was the determination of the category of bodies that have a conical longitudinal contour and a transverse contour consisting of a combination of arcs of a circle and line segments tangential to it, in which all bodies have the same drag. Bodies of this category have been termed absolutely optimum, since, for a specified base area, their drag does not depend on the body length or on the number of cycles of the transversal contour and is determined only by the velocity and the parameters of the medium.

The problem of the three-dimensional shape of a body of minimum drag is solved below without any constraint on the body thickness and without assuming similarity of its cross-sections [1–8], and also without constraints on the type of function prescribing the model of local interaction of the medium with the body surface.

1. FORMULATION OF THE PROBLEM

Let us examine the motion of a body in a medium with a constant velocity in a direction opposite to the direction of a certain axis prescribed by unit vector **x**.

The drag of the body is written in the form

$$D = q\iint_S [c_p(\mathbf{nx}) + c_\tau(\boldsymbol{\tau}\mathbf{x})]dS \tag{1.1}$$

where q is the velocity head, c_p and c_τ are the pressure and friction coefficients on the body surface, **n** and $\boldsymbol{\tau}$ are the unit vectors of the inward normal and the tangent to the surface element, and integration is carried over the surface S of contact between the medium and body, for which

$$\alpha = (\mathbf{nx}) \geqslant 0, \quad (\boldsymbol{\tau}\mathbf{x}) = \sqrt{1-\alpha^2} \tag{1.2}$$

Let each surface element of the body react with the medium independently of the others, and let the coefficients, c_p and c_τ, within the framework of the model of local interaction, be functions of α:

$$c_p = c_p(\alpha), \quad c_\tau = c_\tau(\alpha) \tag{1.3}$$

In the general case, the parameters of the medium and the velocity of the body, which are assumed to be constant, may occur in expressions (1.3). In particular, relations (1.3) include models of force interaction that were used earlier in the formation of the problems in [1–8], with

$$c_p = A_1\alpha^2 + B_1\alpha + C_1, \quad c_\tau = A_2\alpha^2 + B_2\alpha + C_2 \tag{1.4}$$

Here A_i, B_i and C_i (i = 1, 2) are constant parameters of the local model, which depend on the parameters of the medium and the velocity. It has been shown [8] that, with certain assumptions, expressions (1.4) can describe the pressure and friction coefficients on the body surface during its motion in gases and dense media such as the ground and metals

In a cylindrical system of coordinates (ρ, x, θ) with reference scale x at the tip of the body, let the surface of the latter be specified by the equation

$$\rho = \psi(x, \theta)$$

where $\psi(0, \theta) = 0$. It is assumed that the function $\psi(x, \theta)$ is continuous over the entire region of the analysis: $x \in [0, x_k]$, $\theta \in [0, 2\pi]$, where x_k is the body length, and its partial derivatives ψ_x and ψ_θ can undergo a discontinuity on a finite number of break lines. With $x = x_k$, the cross-section area S_b (base area) is considered to be specified, and consequently

$$\int_0^{2\pi} \psi^2(x_k, \theta) d\theta = 2S_b \tag{1.5}$$

For the parameter α and the differential of the area dS it is possible to write the expressions

$$\alpha = \psi_x / \left[1 + (\psi_\theta / \psi)^2 + \psi_x^2\right]^{1/2}, \quad dS = [\psi\psi_x / \alpha] dx d\theta \tag{1.6}$$

The surface of integration in (1.1) should be "wettable", and on it

$$\psi_x \geqslant 0, \quad 1 \geqslant \alpha \geqslant 0 \tag{1.7}$$

Taking relations (1.2), (1.3) and (1.6) into account, expression (1.1) can be written in the form

$$D = q \int_0^{2\pi} d\theta \int_0^{x_k} f(\alpha) \psi\psi_x dx \tag{1.8}$$

The function $f(\alpha)$ in (1.8), which depends on the resistance law (1.3), in general is written as

$$f(\alpha) = c_p(\alpha) + (c_\tau(\alpha)/\alpha)(1 - \alpha^2)^{1/2} \tag{1.9}$$

As a result, the problem of constructing a three-dimensional body of minimum drag can be formulated in the following form: among the functions $\psi(x, \theta)$ and $\alpha(x, \theta)$ which satisfy the conditions of smoothness defined above, and also conditions (1.5) and (1.7) and the connection equation (1.6), it is required to find those which provide the minimum of drag functional (1.8).

2. EXTREMAL SURFACES

The problem of minimizing functional (1.8) can be solved using the general methods of the calculus of variations. For this it is necessary to write the Lagrangian and to obtain Euler's equations for the functions occurring in it. The analytical solution of the corresponding system of non-linear second order partial differential equations, taking condition (1.5) into account, presents great difficulties, and additional constraints must therefore be imposed on the category of permissible surfaces. This was done earlier in [1–8], when a solution was sought in the category of thin bodies with similar cross-sections along the axis of the body. This problem, however, allows of a different approach which does not require any additional constraints on the surfaces required, apart from conditions (1.5) and (1.7) introduced in its formulation.

For a positive argument α, where $\alpha \in [0, 1]$, let us consider the function $f(\alpha)$ from (1.9). Suppose that, for $\alpha = \alpha^* \in [0, 1]$, it reaches a minimum. Then, taking into account the form of the integrand of functional D (1.8), we obtain that, for any surface $\psi(x, \theta)$ and the function $\alpha(x, \theta)$ satisfying conditions (1.5) and (1.7)

$$D \geqslant D^* = qf(\alpha^*) \int_0^{2\pi} d\theta \int_0^{x_k} \psi\psi_x dx = qS_b f(\alpha^*) \tag{2.1}$$

This inequality gives a lower limit of the values of functional (1.8). The equality is possible only when

$\alpha(x, \theta) \equiv \alpha^*$, where the functional D reaches its absolute minimum D^*. Thus, the extremal surfaces in the problem of the shape of a body of minimum drag with a specified base areas S_b are surfaces with $(\mathbf{nx}) \equiv \alpha^* \equiv$ const. As follows from (1.9), the value of α^* for which the function $f(\alpha)$ is a minimum does not depend on S_b and is determined solely the parameters of the medium and the velocity.

In accordance with the expression of α (1.6), the function $\psi(x, \theta)$, giving the shape of extremal surfaces, satisfies the equation

$$\psi_x / [1 + (\psi_\theta / \psi)^2]^{1/2} = t^*, \quad t^* = \alpha^* / (1 - \alpha^{*2})^{1/2} \tag{2.2}$$

When $\alpha^* < 1$, its solutions when $\psi_x = t^*$ and $\psi_\theta = 0$ defined the surfaces of circular cones, in particular a cone with $\psi(x, \theta) = t^* x$ with the origin of coordinates at its vertex. It is obvious that all planes tangential to it also satisfy Eq. (2.2). The value $\alpha^* = 1$ corresponds to the case where the cone degenerates into a plane normal to the x axis. The solution of the problem is then given by any area S_b normal to this axis.

We shall refer to bodies formed by surfaces on which $(\mathbf{nx}) = \alpha^*$ at each point, and with function $\psi(x, \theta)$ satisfying condition (1.5), as "absolutely optimum bodies". All absolutely optimum bodies have an identical drag D^* below which, for a specified base area, S_b it is impossible to go. When $\alpha^* < 1$, this category of bodies includes a circular cone of length $x_k = (S_b/\pi)^{1/2}/t^*$ with $\psi(x, \theta) = t^* x$ and bodies whose surface is formed by planes tangential to it.

The case when $\alpha^* = 0$ in an examination of absolutely optimum bodies is only of theoretical interest, since the length of the corresponding circular cone with a specified based area S_b is infinite. For a finite length, the specified base area S_b of such bodies can be ensured by two methods: by constructing their base from a finite number of rays of infinite length (in the radial direction) or from an infinite number of rays of finite length. Such bodies, being shapes of minimum drag, are not, however, of practical interest. The value $\alpha^* = 0$, in particular, is obtained using Newton's resistance law with zero friction. It is for this reason that other researchers [1, 3, 4], in constructing three-dimensional bodies of minimum wave drag within the scope of Newton's resistance law, along with the length and base area, had to prescribe the minimum radius of the base contour and the number of cycles in it. The setting of these additional constraints enabled shapes of practical interest to be obtained.

3. PROPERTIES OF ABSOLUTELY OPTIMUM BODIES

For the absolutely optimum bodies obtained above, the function $\psi(x, \theta)$ can be written in the form

$$\psi(x, \theta) = \varphi(x) R(\theta) \tag{3.1}$$

with continuous functions $\psi(x)$ and $R(\theta)$, defining, respectively, the longitudinal and transversal contours. We emphasize that here formula (3.1) is the result of solving the problem an question and not the assumption that configurations required are homothetic, as earlier [1–8].

By virtue of the conicity of the longitudinal contour of absolutely optimum bodies

$$\varphi'(x) = t_k = \text{const} > 0 \tag{3.2}$$

In addition, $\psi(x)$ and $R(\theta)$, according to (1.5), satisfy the conditions

$$\varphi(0) = 0, \quad \varphi(x_k) = (S_b / \pi)^{1/2} \tag{3.3}$$

$$R(0) = R(2\pi), \quad \int_0^{2\pi} R^2(\theta) d\theta = 2\pi \tag{3.4}$$

Suppose for a known resistance law and consequently, a known function $f(\alpha)$, the value obtained is $\alpha^* < 1$. Then, as shown above, the body constructed is an absolutely optimum body if $\psi(x, \theta)$ satisfies Eq. (2.2). Taking relations (3.1) and (3.2) into account, Eq. (2.2), after introducing the notation

$$r(\theta) = R^2 / (R^2 + R'^2)^{1/2} \tag{3.5}$$

can be written in the form $t_k r(\theta) = t^*$, Hence, the following expression is obtained for the function $r(\theta)$

$$r(\theta) = r_k = t^* / t_k = \text{const} \tag{3.6}$$

in which the constant r_k depends on the shape of the longitudinal contour. As follows from relations (3.4) and (3.5), $r_k \leqslant 1$.

Thus, an absolutely optimum body has a conical longitudinal contour $\varphi(x) = t_k x$ and a transversal contour with the function $r(\theta) = r_k \leqslant 1$ with constants t_k and r_k related by the equality

$$r_k t_k = t^* \tag{3.7}$$

The functions $\varphi(x)$ and $R(\theta)$ satisfy conditions (3.3) and (3.4). The given category of bodies includes an infinite set of shapes with different longitudinal and transverse contours. Their length x_k, as follows from (3.3), with a prescribed base area S_b, is defined by the expression

$$x_k = (S_b/\pi)^{1/2}/t_k = r_k x_k^*, \quad x_k^* = (S_b/\pi)^{1/2}/t^* \tag{3.8}$$

where x_k^* is the length of the circular cone of the class of absolutely optimum bodies, which depends only on S_b and t^*.

Since $r_k \leqslant 1$, it follows that in all cases $x_k \leqslant x_k^*$, and among absolutely optimum bodies a circular cone has the greatest length. By changing r_k it is possible to produce bodies of different length with longitudinal contour $\varphi(x, \theta) = t_k x$, and the value $t_k \geqslant t^*$ is determined by r_k from expression (3.7).

It t_k and, consequently, the shape of the longitudinal contour of the absolutely optimum bodies are uniquely found from r_k, then the shape of its transversal contour, even for known r_k, is determined non-uniquely. In fact, the function of the transversal contour $R(\theta)$, by relations (3.5) and (3.6), must satisfy the equation

$$R^2/(R^2 + R'^2)^{1/2} = r_k$$

Its solutions are arcs of two types: (1) the arc of a circle with $R(\theta) \equiv r_k$ (a "zero" arc) and (2) the segment of a line tangential to a circle of radius r_k with $R(\theta) = r_k/\cos(\theta - \theta_1)$ (a "positive" arc), where θ_1 is an integration constant. The function $R(\theta)$, continuous in the segment $[0, 2\pi]$, should in this case satisfy conditions (3.4). The presence in the transversal contour of each of the different types of arc depends not only r_k but also on the prescribed number of points of discontinuity of its derivative $R'(\theta)$. As shown below, when $r_k < 1$, such points necessarily exist.

The transversal contour can be made up of N identical cycles, when N is an integer. Each cycle consists of two smooth symmetrical arcs allowing a discontinuity of $R'(\theta)$ at the joining point. Then, for known $r_k \leqslant 1$ and any $N \geqslant 2$, the transversal contour is determined entirely by the function $R(\theta)$ in a half-cycle in the segment $[0, \pi/N]$ and can consist of two arcs smoothly joining at the point $\theta = \theta_0$: (1) a zero arc with $\theta \in [0, \theta_0]$ and (2) a positive arc with $\theta \in [\theta_0, \pi/N]$. The values of θ_0 and $\theta_1$are found from the relations

$$R_0 = \frac{r_k}{\cos(\theta_0 - \theta_1)}, \quad \operatorname{tg}(\theta_0 - \theta_1) + \frac{NR_0^2}{\pi - N\theta_0 R_0^2} = \operatorname{ctg}\left(\frac{\pi}{N} - \theta_0\right) \tag{3.9}$$

where R_0 is the minimum value of $R(\theta)$, and here, without loss of generality, $R_0 = R(0)$. If

$$r_k \leqslant R_n, \quad R_n = [(\pi/N)/\operatorname{tg}(\pi/N)]^{1/2} \tag{3.10}$$

then there is no zero arc for the transversal contour ($\theta_0 = 0$). If condition (3.10) is not satisfied and $r_k < 1$, the contour contains both positive and zero arcs. In this case $\theta_1 = \theta_0$, and $R_0 = r_k$.

As an example, Fig. 1 shows transversal contours 1–3 for $N = 4$ and $r_k = 1.095$ and 0.5. The functions $R(\theta)$ of these contours satisfy conditions (3.4). The dashed lines give circles 4 and 5 of radii $r_k = 0.95$ and 0.5. Segments of lines tangential to them form positive arcs of contours 2 and 3. Since, when $N = 4$, the value $R_n = 0.89$, then, for contour 3 with $r_k = 0.5$, by condition (3.10), there is no zero arc, whereas contour 2 with $r_k = 0.95$ contains both positive and zero arcs. Transverse contours 1–3, mapped in Fig. 1, belong to absolutely optimum bodies if their longitudinal contours $\varphi(x) = t_k x$ have $t_k = t^*/r_k$. This, in particular [see relation (3.8)], leads to an absolutely optimum body with transverse contour 3 ($r_k = 0.5$) having half the length of a circular cone of this category with contour 1.

As follows from relations (3.9), for fixed r_k, by varying the values of N, it is possible to produce transversal contours with different minimum radii R_0. An increase in the number of cycles N leads to an increase in R_0, which enables the transverse dimensions of the body to be varied. This possibility is demonstrated by Fig. 2. The continuous lines represent transversal contours 2 and 3 constructed with

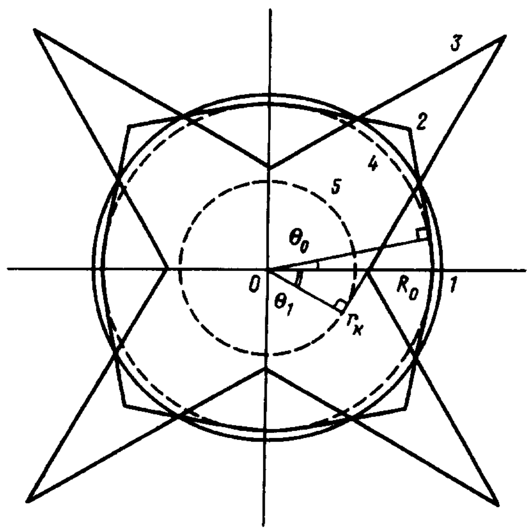

Fig. 1.

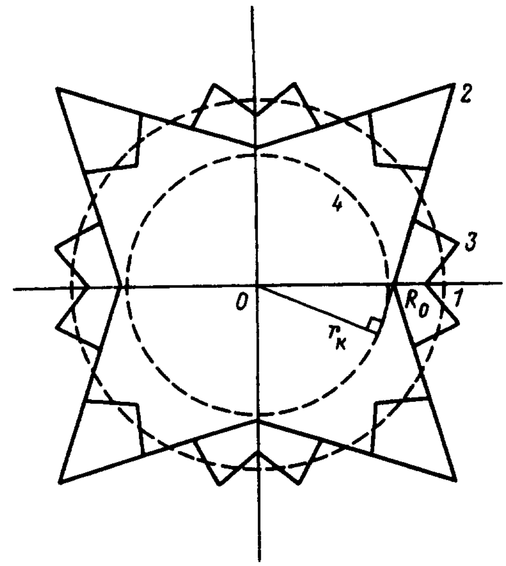

Fig. 2.

$r_k = 0.7$ and $N = 4$ and 16. These contours bound the same area as circle 1, which corresponds to the transversal contour of the circular cone with $r_k = 1$. Circle 4 in Fig. 2 has a radius $r_k = 0.7$, and there are lines tangential to it whose segments form contours 2 and 3. By fixing r_k and increasing N, it is possible to make the radius R_0 differ as little as desired from unity. Here, since all these contours have the same value of r_k, the absolutely optimum bodies corresponding to it have the same longitudinal contour with $\varphi(x) = t_k x$ with t_k from (3.7).

Thus, for a known value of t^* which depends only on the parameters of the medium and the velocity, it is possible to construct an infinite set of absolutely optimum bodies having a conical longitudinal contour with function $\varphi(x) = t_k x$ and a transverse contour which, in general, consists of arcs of a circle and segments of lines with the function $r(\theta) = r_k$, and here t_k and r_k are related by condition (3.7). For specified base area S_b, all these bodies have the same drag D^* (2.1) and gives a solution to the problem of the shape of a body of minimum drag.

The given category of bodies with prescribed S_b includes bodies of different length and different cross-sectional dimensions, which can change continuously without chaining the drag of the body. This property of absolutely optimum bodies can be used in cases where, besides the specification of S_b, additional constraints are imposed on the body required. If, with such constraints, it is possible to select the absolutely optimum body which satisfies them, then, giving the drag functional (1.8) an absolute minimum (2.1), it will be a body of minimum drag in the problem with given constraints.

4. OPTIMUM SHAPES WHEN ADDITIONAL CONSTRAINTS ON THE GEOMETRY OF THE BODY ARE SPECIFIED

Suppose the base area S_b is specified and, from the known parameters of the medium and the velocity, the value $\alpha = \alpha^*$ is obtained for which the function $f(\alpha)$ (1.9) in the segment [0, 1] reaches a minimum. Let $\alpha^* \in (0, 1)$ and from it, from (2.2), the value t^* is determined.

Lemma 1. If, as an additional condition, the maximum permissible (limiting) length of the body L is specified, then an absolutely optimum body always exists which yields a solution of the problem of the shape of a body of minimum drag and which gives the drag functional an absolute minimum.

Proof. From the specified S_b and L we find $t = (S_b/\pi)^{1/2}/L$. We introduce the notation $t_m = \max(t, t^*)$. Any absolutely optimum body with $\varphi(x) = t_k x$, where $t_k \geqslant t_m$, and with $r_k = t^*/t_k$ will be the solution of the problem, since, according to (3.8), its length $x_k \leqslant L$. Here, if $t_k > t^*$, then the body has a star-shaped cross-section consisting of an arbitrary number N of symmetrical cycles, and here, if condition (3.10) is satisfied, there is no zero arc at the transversal contour.

Lemma 2. If, as additional constraints on the geometry of the body, the length L and the characteristic cross-sectional dimension of the base of the body are specified, then an absolutely optimum body always exists which solves the problem of the shape of a body of minimum drag and which gives the drag functional an absolute minimum.

Proof. We will take as the specified cross-sectional dimension of the body, the radius of the cross-section of its base $R_0 < 1$. By relations (3.9) and (3.10), there is always an integer N^* such that, for all $N \geqslant N^*$, the condition $R_n > R_0$ will be satisfied, which means that, for all N, there is no zero arc at the optimum contour. Then, for these N, the value of r_k will be found from the expression

$$r_k = R_0/[1+\mathrm{tg}^2(\theta_1)]^{1/2}, \quad \mathrm{tg}(\theta_1) = \left(R_0^2 - R_n^2\right)/(\pi/N) \tag{4.1}$$

Since, as $N \to \infty$, the value $r_k \to 0$, then N, r_k and $t_k = t^*/r_k$ will always be found such that the conditions $t_k > (S_b/\pi)^{1/2}/L$ and $x_k \leqslant L$ are satisfied. Consequently, there will always be an absolutely optimum body that is admissible in the problem and that will be its solution.

The following is proved by a similar method.

Lemma 3. If, as additional constraints on the geometry of the body, the number of cycles of the transversal contour of the body and also its limiting length or the characteristic cross-sectional dimension of the base are specified, then there is always an absolutely optimum body that yields the solution of the problem of the shape of a body of minimum drag and gives the drag functional an absolute minimum.

Lemmas 1–3 were proved taking into account the condition $\alpha^* < 1$, where $t^* < \infty$, and the case of non-degenerate extremal surface is realized. However, they also remain valid when $\alpha^* = 1$. In this case, the solution is given by any area with specified S_b with the normal directed along the flow velocity vector.

The consequence of Lemmas 1–3 is as follows.

Theorem. If, for a specified base area S_b, there is at least one arbitrariness in specifying the limiting length of the body L, the characteristic cross-sectional dimension of the base R_b or the number of cycles N of the transversal contour, and there are no other constraints on the geometry of the body, then there is always an absolutely optimum body that yields a solution of the problem of the shape of a body of minimum drag and gives the drag functional (1.8) an absolute minimum (2.1).

When arbitrary values of S_b, L, R_b and N are simultaneously specified, there may not be an absolutely optimum body that satisfies all the specified conditions. A similar situation arises when, for specified S_b and N, values $L < x_k^*$ are adopted, where x_k^* is the length of the cone from the body category (3.8), and the minimum values of the radius of the base $R_b > R_0$, where R_0, for specified N, is found from relation (3.9) with $r_k = t^*/t$, where $t = (S_b/\pi)^{1/2}/L$. In such cases, the optimum configurations have a more complex shape and their construction requires the simplifications and approaches used earlier [7, 8]. In particular, as shown in [8] for thin bodies with resistance law (1.4), the optimum longitudinal contour is then not rectilinear and, at the tip of the body, is approximated by a power law function with an exponent of 3/4.

When using the results of earlier investigations [7, 8], where all of the above constraints on the geometry of the body were included in the final formulation of the problem, it is important to avoid the errors and inaccuracies they contain.

Without making a detailed examination of the solution given in [7], we shall merely note that, in writing Euler's equation in this paper for the function of the longitudinal contour, an error occurred, and, in searching for the optimum transversal contour, the incorrect conclusion was reached that, for specified arbitrary values of S_b, L and R_0 and a variation in the parameter N, no solution of the problem exists. However, as follows from the proof of Lemma 2, an infinite set of absolutely optimum body shapes exist that in this case are the solution of the problem, with lengths $x_k \leqslant L$, which, where necessary, can always be brought up to the prescribed length L by a needle of zero thickness in front of the body.

In solving the problem of a three-dimensional shape of minimum drag in the same formulation as in [7] but using, for the forces acting on a surface element of the body, relations (1.4) [8], again a number of inaccurate conclusions were reached in relation to the optimum three-dimensional shape. In particular, it must be pointed out that, for specified S_b, L, R_0 and N, the optimum transversal contour cannot contain a non-zero convex arc if there are no additional constraints on the longitudinal contour. A regular convex arc in the solution given in [8] appears for values of $R_0 < t^*/t$, where $t = (S_b/\pi)^{1/2}/L$. However, in this case there is always an absolutely optimum body with a transversal contour of N symmetrical cycles and with a value of $r_k = R_0$ if $R_0 \geqslant R_n$, and r_k, $< R_0$ if $R_0 < R_n$, and then r_k is found from R_0 and R_n from expression (4.1), Since for this body $t_k = t^*/r_k > t$, its length (3.8) $x_k \leqslant L$, which, where necessary, can be brought to the prescribed length L by a needle of zero thickness. A convex arc at the optimum transversal contour in the formulation of the problem [8] is possible only with an additional constraint on the longitudinal contour, when, for example, it is assumed to be specified.

Above, it was assumed that the transversal contour of the body consists of N identical symmetrical cycles. In fact, the class of absolutely optimum bodies is much wider. It obviously includes any configuration with a conical longitudinal contour with the function $\varphi(x) = t_k x$, where $t_k \geqslant t^*$, and a transversal contour $R(\theta)$, which, satisfying conditions (3.4), is made up of any combinations of arcs of a circle of radius $r_k = t^*/t_k$ and segments of lines tangential to it. It can be shown that the transversal contours of asymmetrical absolutely optimum bodies, like symmetrical ones, cannot contain arcs of a circle.

5. A SPECIAL CASE OF A FORCE EFFECT OF THE MEDIUM ON A SURFACE ELEMENT OF THE BODY

Let us consider the specified case of the solution $f(\alpha)$, when the force effect of the medium on a surface element of the body can be described by relations (1.4) with $B_1 = B_2 = 0$. The given model of force interaction covers not only hypersonic motion of the body in a gas with a constant coefficient of friction ($A_1 \neq 0$, $C_2 \neq 0$ and $C_1 = A_2 = 0$) but also high-speed motion of the body in dense media such as the ground or metals, when, with $A_2 = \mu_0 A_1$ and $C_2 = \mu_0 C_1$, a model of motion with Coulomb friction holds (μ_0 is the coefficient of dry friction) and, with $A_1 \neq 0$, $C_1 \neq 0$, $C_2 \neq$ and $A_2 = 0$, the model of motion with constant friction holds.

In the case in question

$$f(\alpha) = A_1\alpha^2 + (A_2\alpha + C_2/\alpha)\gamma + C_1, \quad \gamma = (1-\alpha^2)^{1/2} \tag{5.1}$$

A local extremum of this function is reached with a value of α defined as the root of the equation

$$f'(\alpha) = \alpha^2(2\alpha\gamma + A(1-2\alpha^2)) - C = 0, \quad A = A_2/A_1, \quad C = C_2/A_1 \tag{5.2}$$

The global minimum of the function $f(\alpha)$ in the segment [0, 1] is sought among its local minimum and boundary minima (possible with $\alpha = 0$ and $\alpha = 1$).

For the model with constant friction ($A = 0$, $C \neq 0$), the solution of Eq. (5.2) is determined by the value of the single parameter $Y = C^{1/3}$ and, in the case of a thin body ($\alpha^2 \ll 1$) has the form

$$\alpha^* = 2^{-1/3}Y \tag{5.3}$$

In the thin body approximation, solution (5.3) with $Y \leqslant 2^{1/3}$ defines the global minimum of the function $f(\alpha)$ in a segment [0, 1], which for $Y \geqslant 2^{1/3}$, is reached at the point $\alpha = 1$, where the assumption that the body is thin no longer holds. The dependence of α^* on Y (5.3) is shown in Fig. 3 (curve 1). For

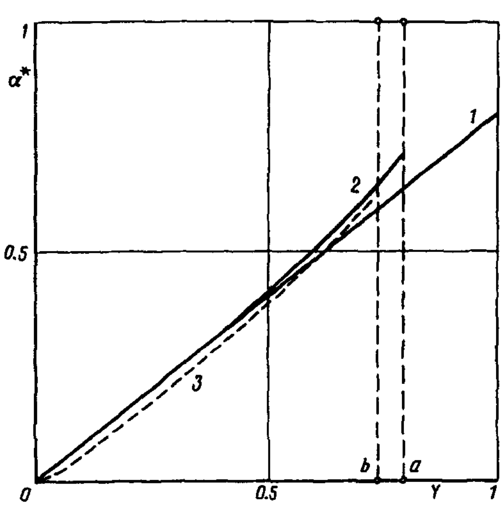

Fig. 3.

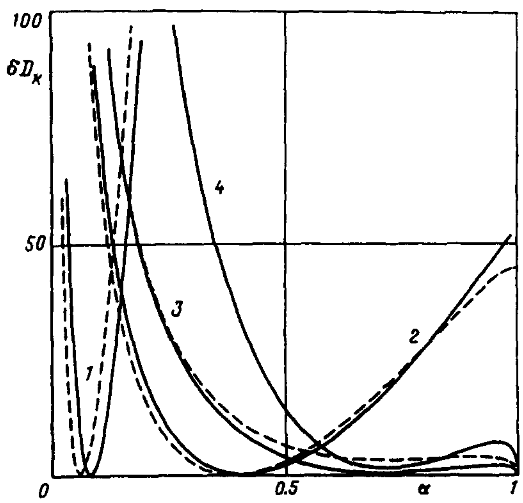

Fig. 4.

$Y < 0.5$ it is similar to the exact solutions of the problem of minimum of the function $f(\alpha)$, found without any constraint on the body thickness. In Fig. 3, the exact solutions are represented by curve 2 (for the model of constant friction with $A = 0$) and by curve 3 (for the model of Coulomb friction with $A = \mu_0 = 0.2$). Points *a* and *b* for curves 2 and 3 mark the values of Y^* for which the local and boundary minima (with $\alpha = 1$) of the function $f(\alpha)$ and identical. When $Y > Y^*$, the minimum is reached at the point $\alpha = 1$.

For different Y, the drag of absolutely optimum bodies, D^*, was compared with the drag of circular cones, D_k, with the same base area. Figure 4 gives, as a function of α, values of $\delta D_k = (D_k/D^* - 1) \times 100$ indicating that the drag of a cone, D_k, with an aperture half-angle $\beta = \arcsin \alpha$ exceeds the value of D^*. Curves 1–3 were plotted for $Y = 0.1$, 0.5 and 0.8, respectively. The continuous curves in Fig. 4 correspond to the model of constant friction with $C_1/A_1 = 5Y^3$, and the dashed curves correspond to the model of Coulomb friction with $\mu_0 = 0.2$. As an example, Fig. 4 gives curve 4 for $C_1 = 0$, plotted for the model of constant friction with $Y = 0.8$.

Since $D_k = qf(\alpha)S_b$, and the drag of absolutely optimum bodies, D^*, is defined by expression (2.1), it follows that curves 1–4 in Fig. 4, besides δD_k with the same Y, demonstrate the behaviour of the function $(f(\alpha)/f(\alpha^*) - 1)$ as a function of α. In particular, curves 3 and 4 show the behaviour of this function in the segment [0,1] with $Y = 0.8$, when $\alpha^* = 1$, and for both friction models the minimum of the function $f(\alpha)$ changed from local to boundary. Thus, for the chosen drag law, whatever the value of Y, there is always a region of α values in which absolutely optimum bodies have a significantly lower drag compared with cones of the same length and the same base area with $\beta = \arcsin \alpha$.

6. CONCLUSION

Within the framework of the law of locality, without constraints on the body thickness or the type of function specifying the interaction model, bodies have been constructed that are called "absolutely optimum bodies". These bodies, with different configurations, have an identical drag below which, for specified base area, it is impossible to go. They are conical. Their surface is formed by sections of a circular cone, which may be absent, and sections of planes tangential to it. The angle at the tip of the circular cone is determined by the characteristics of the medium and the velocity in terms of constants which occur in the drag law. Absolutely optimum bodies can be 'star-shaped' with transversal contours made up of an integer number of symmetrical cycles. A theorem has been proved with an enumeration of the constraints on the geometry of the body for which absolutely optimum bodies are solutions of the problem of the shape of a body of minimum drag. In particular. absolutely optimum bodies will always be the solution of such problem for a specified length and base area of the sought body required. For a particular form of the drag law, including Newton's law of resistance with constant friction, made the drags of absolutely optimum bodies and of circular cones of equivalent length and base area have been compared. It has been shown that, for a known friction parameter, there will always be relative thicknesses of the body for which the drag of the absolutely optimum body is considerably lower.

The results obtained agree with the main conclusions of [9, 10] that a considerable reduction in aerodynamic drag in the class of bodies of equivalent length and base area can be achieved by changing from axisymmetrical bodies to star-shaped ones. In fact, for a gas, the local coefficient of friction at hypersonic speeds is of the order of 10^{-2}–10^{-3}. Assuming it to be constant ($Y \approx 0.1$), from relation (5.3) we find $\alpha^* \approx 0.1$. In this case, the aperture half-angle of a circular cone in the also of absolutely optimum bodies is $\beta^* \approx 5°$, and its relative thickness is $2t^* \approx 0.2$. If $t = (S_b/\pi)^{1/2}/L > t^*$, then, as shown above, for specified S_b and L, the solution of the problem of the body with minimum drag will be an absolutely optimum body with a star-shaped cross-section. Earlier [9, 10], theoretical and experimental investigations were made of the aerodynamic characteristics of pyramidal bodies with plane faces with $t = 0.2$–0.4. Curve 1 in Fig. 4, plotted for constant friction with $Y = 0.1$. (the continuous curve) and $\alpha^* = 0.08$, gives in this case the drag of a cone with an apertuse half-angle of $\beta = \arcsin \alpha$ that exceeds $D = D^*$ of an absolutely optimum body. Consequently, as in earlier papers [9, 10], for these values of t, the drag of star-shaped absolutely optimum bodies is considerably lower than the drag of equivalent circular cones.

In the problems examined here, the absolutely optimum bodies turned out to be conical. This must be borne in mind when analysing the results of recently published work [11], where the aerodynamic characteristics of star-shaped bodies, the longitudinal contours of which with reference to [4] were chosen to be exponential ($\varphi(x) = kx^n$ with $n = 3/4$), were determined numerically and experimentally. However, are recall that the optimum value $n = 3/4$ was obtained [4] for specified S_b, L, N and R_0. whereas the characteristics of starbodies with exponential and conical longitudinal contours were compared [11] for specified L and volume.

In conclusion, we note that review [12] of the results of theoretical and experimental research into the reduction in drag of starbodies when they move in dense media such as the ground or metals showed that, for penetration velocities of the order of 10^2–10^3 m/s, approximate models of type (1.4) can be used to record the stresses on the surface of three-dimensional bodies. This confirms the possibility of also using the results of the present paper to optimize the shape of three-dimensional bodies when they move through dense media.

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